$$\int \sqrt{\mathbf{1+x^2}}\,\boldsymbol{dx}$$

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Our aim is to prove $I = \int \sqrt{1+x^2}\ dx = \frac{1}{2}\left[x\sqrt{1+x^2} + \ln\left|x+\sqrt{1+x^2}\right|\right] + C$ using different methods.

Method 1 (Trigonometric substitution)

Let $x = \tan\theta$, $x = \sec^2\theta\, d\theta$, $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$

$\int \sqrt{1+x^2}\ dx = \int \sec\theta\,(\sec^2\theta\, d\theta) = \int \sec^3\theta\, d\theta$

We therefore like to evaluate $I = \int \sec^3\theta\, d\theta$.

Using integration by parts,

$$I = \int \sec^3\theta\, d\theta = \int \sec\theta\, d(\tan\theta) = \tan\theta\sec\theta - \int \tan\theta\, d(\sec\theta)$$

$$= \tan\theta\sec\theta - \int \tan\theta\,(\sec\theta\tan\theta)d\theta = \tan\theta\sec\theta - \int \sec\theta\tan^2\theta\, d\theta$$

$$= \tan\theta\sec\theta - \int \sec\theta\,(\sec^2\theta - 1)\, d\theta$$

$$= \tan\theta\sec\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

$$2I = \tan\theta\sec\theta + \int \sec\theta\, d\theta = \tan\theta\sec\theta + \int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta} d\theta$$

$$= \tan\theta\sec\theta + \int \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta+\tan\theta} d\theta = \tan\theta\sec\theta + \int \frac{d(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}$$

$$\therefore I = \frac{1}{2}[\tan\theta\sec\theta + ln|\sec\theta+\tan\theta|] + C = \frac{1}{2}\left[x\sqrt{1+x^2} + \ln\left|x+\sqrt{1+x^2}\right|\right] + C$$

$$\therefore \int \sqrt{1+x^2}\ dx = \frac{1}{2}\left[x\sqrt{1+x^2} + \ln\left|x+\sqrt{1+x^2}\right|\right] + C$$

Method 2 (Using two variables)

Let $y^2 = 1 + x^2 \Longrightarrow ydy = xdx$

Consider $I = \int \sqrt{1+x^2}\ dx = \int ydx = xy - \int xdy$

Hence $I = \frac{1}{2}[xy + \int (ydx - xdy)] = \frac{1}{2}\left[xy + \int \frac{(x+y)(ydx-xdy)}{x+y}\right] = \frac{1}{2}\left[xy + \int \frac{xydx+y^2dx-x^2dy-xydy}{x+y}\right]$

$$= \frac{1}{2}\left[xy + \int \frac{y(ydy)+y^2dx-x^2dy-x(xdx)}{x+y}\right] = \frac{1}{2}\left[xy + \int \frac{y(ydy)+y^2dx-x^2dy-x(xdx)}{x+y}\right]$$

$$= \frac{1}{2}\left[xy + \int \frac{(y^2-x^2)dx+(y^2-x^2)dy}{x+y}\right] = \frac{1}{2}\left[xy + \int \frac{d(x+y)}{x+y}\right] = \frac{1}{2}[xy + \ln|x+y|] + C$$

$$= \frac{1}{2}\left[x\sqrt{1+x^2} + \ln\left|x+\sqrt{1+x^2}\right|\right] + C$$

Method 3 (Hyperbolic substitution)

Put $x = \sinh\theta \Longrightarrow dx = \cosh\theta\, d\theta$

$\sqrt{1+x^2} = \sqrt{1+\sinh^2\theta} = \cosh\theta$

$$I = \int \sqrt{1+x^2}\; dx = \int \cosh\theta\,(\cosh\theta\, d\theta) = \int \cosh^2\theta\, d\theta$$

$$= \frac{1}{2}\int (1+\cosh 2\theta)\, d\theta = \frac{1}{2}\left[\theta + \frac{\sinh 2\theta}{2}\right] = \frac{1}{2}\left[\theta + \frac{2\sinh\theta\cosh\theta}{2}\right] = \frac{1}{2}\left[\sinh\theta\sqrt{1+\sinh^2\theta} + \theta\right]$$

$$= \frac{1}{2}\left[x\sqrt{1+x^2} + \sinh^{-1}x\right] + C$$

Proof: $\sinh^{-1}x = \ln\left|x+\sqrt{1+x^2}\right|$

By definition of $x = \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2} = \frac{1}{e^{\theta}}\left(e^{2\theta}-1\right)$

$$e^{2\theta} - xe^{\theta} - 1 = 0$$

By quadratic equation formula, $e^{\theta} = x + \sqrt{1+x^2}$, taking the positive root.

$$\theta = \sinh^{-1}x = \ln\left|x+\sqrt{1+x^2}\right|$$

Method 4 (Variant of hyperbolic substitution)

$I = \int \sqrt{1+x^2}\; dx$

Let $x = \frac{1}{2}\left(t - \frac{1}{t}\right) \Longrightarrow dx = \frac{1}{2}\left(1+\frac{1}{t^2}\right) dt$

Also, $1 + x^2 = \frac{1}{4}\left(t^2 + 2 + \frac{1}{t^2}\right) \Longrightarrow \sqrt{1+x^2} = \frac{1}{2}\left(t+\frac{1}{t}\right)$

$$\therefore I = \int \frac{1}{2}\left(t+\frac{1}{t}\right)\left[\frac{1}{2}\left(1+\frac{1}{t^2}\right)dt\right] = \frac{1}{4}\int\left[t + \frac{1}{t^3} + \frac{2}{t}\right]dt = \frac{1}{4}\left[t^2 - \frac{1}{t^2} + 2ln|t|\right]$$

$$= \left[\frac{1}{2}\left(t-\frac{1}{t}\right)\right]\left[\frac{1}{2}\left(t+\frac{1}{t}\right)\right] + \frac{1}{2}ln|t|$$

Note that $x + \sqrt{1+x^2} = \frac{1}{2}\left(t-\frac{1}{t}\right) + \frac{1}{2}\left(t+\frac{1}{t}\right) = t$

$$\therefore I = \frac{1}{2}\left[x\sqrt{1+x^2} + \ln\left|x+\sqrt{1+x^2}\right|\right] + C$$

Method 5

Can we use the integration by parts at the very beginning?

$I = \int \sqrt{1+x^2}\, dx$

Let $u = \sqrt{1+x^2},\; du = \frac{x}{\sqrt{1+x^2}}dx$

$dv = dx, v = x$

$$I = \int \sqrt{1+x^2}\, dx = uv - \int v du = x\sqrt{1+x^2} - \int x\left(\frac{x}{\sqrt{1+x^2}}dx\right) = x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}}dx$$

$$= x\sqrt{1+x^2} - \int \frac{(1+x^2)-1}{\sqrt{1+x^2}}dx = x\sqrt{1+x^2} - \left(\int \sqrt{1+x^2}\, dx - \int \frac{1}{\sqrt{1+x^2}}dx\right)$$

$$= x\sqrt{1+x^2} - I + \int \frac{1}{\sqrt{1+x^2}} dx$$

$$2I = x\sqrt{1+x^2} + \int \frac{1}{\sqrt{1+x^2}} dx \Rightarrow I = \frac{1}{2}\left(x\sqrt{1+x^2} + \int \frac{1}{\sqrt{1+x^2}} dx\right)$$

It leaves to prove $J = \int \frac{1}{\sqrt{1+x^2}} dx = \ln\left|x + \sqrt{1+x^2}\right| + C$

There are various ways, such as using trigonometric or hyperbolic substitutions.
Let $y^2 = 1 + x^2$

$$\therefore ydy = xdx \Rightarrow \frac{dx}{y} = \frac{dy}{x} \Rightarrow \frac{dx}{y} = \frac{dy}{x} = \frac{dx+dy}{y+x} = \frac{d(x+y)}{x+y} \quad \text{(using Compoendo in proportion)}$$

$$\therefore J = \int \frac{1}{\sqrt{1+x^2}} dx = \int \frac{dx}{y} = \int \frac{d(x+y)}{x+y} = ln|x+y| + C = \ln\left|x + \sqrt{1+x^2}\right| + C$$

Method 6 Euler substitution

Those who are unfamiliar with Euler substitution can search in the internet.

(i) $I = \int \sqrt{1+x^2}\ dx$

Let $\sqrt{1+x^2} = x + t \Rightarrow 1 + x^2 = x^2 + 2xt + t^2 \Rightarrow x = \frac{1-t^2}{2t}$

$$\sqrt{1+x^2} = \frac{1-t^2}{2t} + t = \frac{t^2+1}{2t}$$

$$dx = -\frac{t^2+1}{2t^2} dt$$

$$I = \int \frac{t^2+1}{2t}\left(-\frac{t^2+1}{2t^2} dt\right) = -\frac{1}{4}\int \frac{t^4+2t^2+1}{t^3} dt = -\frac{1}{4}\int \left(t + \frac{1}{t^3} + \frac{2}{t}\right) dt$$

$$= -\frac{1}{4}\left[\frac{t^2}{2} - \frac{1}{2t^2} + 2ln|t|\right] = -\frac{1}{4}\left[\frac{t^4-1}{2t^2} + 2ln|t|\right] = -\frac{1}{4}\left[\frac{(t^2-1)(t^2+1)}{2t^2} + 2ln|t|\right]$$

$$= \frac{1}{2}\left[\left(\frac{1-t^2}{2t}\right)\left(\frac{t^2+1}{2t}\right) + ln|t|\right] = \frac{1}{2}\left[x\sqrt{1+x^2} + \ln\left|x + \sqrt{1+x^2}\right|\right] + C$$

(ii) If you study more closely, **Method 4** is exactly the same as the Euler substitution:
$\sqrt{1+x^2} = -x + t$. Try yourselves.

Exercise

Prove that $\int \sqrt{1-x^2}\ dx = \frac{1}{2}\left[x\sqrt{1+x^2} + \sin^{-1} x\right] + C$